On Clifford Subalgebras, Spacetime Splittings and Applications

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Abstract

$\mathbb{Z}_2$-gradings of Clifford algebras are reviewed and we shall be concerned with an α-grading based on the structure of inner automorphisms, which is closely related to the spacetime splitting, if we consider the standard conjugation map automorphism by an arbitrary, but fixed, splitting vector. After briefly sketching the orthogonal and parallel components of products of differential forms, where we introduce the parallel [orthogonal] part as the space [time] component, we provide a detailed exposition of the Dirac operator splitting and we show how the differential operator parallel and orthogonal components are related to the Lie derivative along the splitting vector and the angular momentum splitting bivector. We also introduce multivectorial-induced α-gradings and present the Dirac equation in terms of the spacetime splitting, where the Dirac spinor field is shown to be a direct sum of two quaternions. We point out some possible physical applications of the formalism developed.

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1 Introduction

The class of $\mathbb{Z}_2$-gradings have a paramount importance in the investigations concerning, e.g., generalized N-extended supersymmetries, since this formalism needs a graded Lie algebra to be constructed. On the other hand, the relationship between spacetime splitting and Physics have been investigated in an wide context, e.g., Arnowitt-Deser-Misner (ADM) formalism [1], and the splitting of the group SL(2,$\mathbb{O}$) into the Lorentz group SL(2,$\mathbb{C}$). The main purpose of this paper is to give a complete characterization concerning splitting in Clifford algebra, with the purpose of using $\mathbb{Z}_2$-gradings in order to implement spacetime splittings in Clifford algebras, and to apply this formalism in order to obtain the most general form of Dirac equation from the inner automorphic α-grading viewpoint.

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After presenting algebraic preliminaries on Clifford algebras in Sec. (2), we revisit in Sec. (3) the formalism of $\mathbb{Z}_2$-gradings of Clifford algebras [11]. In Sec. (4) spacetime splitting naturally arises from the inner automorphism framework via a particular chosen α-grading, where we show some properties of the orthogonal [temporal] and parallel [spatial] components of the Clifford product. After presenting the metric splitting in Sec. (5), in Sec. (6) the splitting of the dual Hodge star operator is studied, and in Sec. (7) the Dirac operator is investigated from the point of view of the parallel and orthogonal components of differential and codifferential operators. We show how these components are related to the Lie derivative along the 1-form field n and to the angular momentum 2-form field. Besides, we completely investigate decompositions of the codifferential operator and also of the Hodge dual operator in terms of the proposed α-grading. In Secs. (8) and (9) the covariant and Lie derivatives associated with the differential and codifferential operators are respectively derived in the context of the inner automorphic α-grading. In Sec. (10) we exhibit a dual equivalent inner automorphic α-grading. Relatively to the given α-grading $\alpha(\psi) = n\hat{\psi}n^{-1}$, responsible for the spacetime splitting in successive space-like manifold slices which are orthogonal to an observer worldline, there exists an equivalent dual decomposition in terms of the volume element associated with the α-even subalgebra. The Dirac operator is calculated in each one of these two equivalent dual splittings. In Sec. (11) multivectorial inner automorphisms are introduced, generalizing the (1-)vetorial inner automorphisms presented, and we also exhibit all possible splittings of elements in $\mathcal{C}\ell_{1,3}$. Finally in Sec. (12) Dirac equation splitting is presented in a straightforward manner. After briefly revisiting Hestenes approach of Dirac theory [3, 4] — where the usual even/odd splitting is used to perform spacetime splittings (called *projective splitting*) — the Dirac equation associated with any α-grading [11] is investigated for 1-form fields $n = n(x)$, from the α-grading $\alpha(\psi) = n\psi n^{-1}$ viewpoint. We show that the spinor field satisfying Dirac equation is described by the sum of two quaternions, besides writing Dirac equation in this context. We compare our result with the previous results in the literature, based on the α-gradings induced by standard, chiral (Weyl) and Majorana idempotents representations of $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ in $\mathcal{M}(4,\mathbb{C})$ [11].

2 Preliminaries

Let V be a finite n-dimensional real vector space. We consider the tensor algebra $\bigoplus_{i=0}^{\infty} T^i(V)$ from which we restrict our attention to the space $\Lambda(V) = \bigoplus_{k=0}^{n}\Lambda^k(V)$ of multivectors over V. $\Lambda^k(V)$ denotes the space of the antisymmetric k-tensors, isomorphic to the k-forms. Given $\psi \in \Lambda(V)$, $\tilde{\psi}$ denotes the *reversion*, an algebra antiautomorphism given by $\tilde{\psi} = (-1)^{[k/2]}\psi$ ($[k]$ denotes the integer part of k). $\hat{\psi}$ denotes the *main automorphism or graded involution*, given by $\hat{\psi} = (-1)^k\psi$. The *conjugation* is defined as the reversion followed by the main automorphism. If V is endowed with a non-degenerate, symmetric, bilinear map $g : V \times V \to \mathbb{R}$, it is possible to extend g to $\Lambda(V)$. Given $\psi = \mathbf{u}_1 \wedge\cdots\wedge \mathbf{u}_k$ and $\phi = \mathbf{v}_1 \wedge\cdots\wedge \mathbf{v}_l$, $\mathbf{u}_i, \mathbf{v}_j \in V$, one defines $g(\psi, \phi) = \det(g(\mathbf{u}_i, \mathbf{v}_j))$ if $k = l$ and $g(\psi, \phi) = 0$ if $k \neq l$. Finally, the projection of a multivector $\psi = \psi_0+\psi_1+\cdots+\psi_n$, $\psi_k \in \Lambda^k(V)$, on its p-vector part is given by $\langle\psi\rangle_p = \psi_p$. The Clifford product between

$\mathbf{w} \in V$ and $\psi \in \Lambda(V)$ is given by $\mathbf{w}\psi = \mathbf{w} \wedge \psi + \mathbf{w} \lrcorner \psi$. The Grassmann algebra $(\Lambda(V), g)$ endowed with the Clifford product is denoted by $\mathcal{C}\ell(V, g)$ or $\mathcal{C}\ell_{p,q}$, the Clifford algebra associated with $V \simeq \mathbb{R}^{p,q}$, $p + q = n$. In what follows $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ denote respectively the real, complex and quaternionic (scalar) fields.

3 α-projections

A vector space V is said to be graded by an abelian group G if it is expressible as a direct sum $V = \bigoplus_i V_i$ of subspaces labeled by elements $i \in G$. We say that the Clifford algebra $\mathcal{C}\ell(V, g)$ is graded by G if its underlying vector space is a G-graded vector space and its product satisfies $\deg(ab) = \deg(a) + \deg(b)$. Heretofore we consider $G = \mathbb{Z}_2$. The usual $\mathbb{Z}_2$-grading of $\mathcal{C}\ell(V, g)$ is given by $\mathcal{C}\ell(V, g) = \mathcal{C}\ell^+(V, g) \oplus \mathcal{C}\ell^-(V, g)$, where $\mathcal{C}\ell^{+(-)}(V, g)$ denotes the sum of all the even (odd) k-vectors. An arbitrary $\mathbb{Z}_2$-grading of $\mathcal{C}\ell(V, g)$ is given [11] by $\mathcal{C}\ell(V, g) = \mathcal{C}\ell_0 \oplus \mathcal{C}\ell_1$, where the subspaces $\mathcal{C}\ell_i, i = 0, 1$, satisfy $\mathcal{C}\ell_i \mathcal{C}\ell_j \subseteq \mathcal{C}\ell_{i+j(\text{mod } 2)}$. To each decomposition there is an associated vector space automorphism[1] $\alpha : \mathcal{C}\ell(V, g) \to \mathcal{C}\ell(V, g)$ defined by $\alpha|_{\mathcal{C}\ell_i} = (-1)^i \mathrm{id}_{\mathcal{C}\ell_i}$. The projections π_i on $\mathcal{C}\ell_i$ are given by

$$\pi_i(\psi) = \frac{1}{2}(\psi + (-1)^i \alpha(\psi)). \tag{1}$$

In the sequel we denote $\pi_i(\psi) = \psi_i$. An element that belongs to $\mathcal{C}\ell_0$ ($\mathcal{C}\ell_1$) is called *α-even* (*α-odd*). With respect to the usual $\mathbb{Z}_2$-grading, $\mathcal{C}\ell_0 = \mathcal{C}\ell^+_{p,q}$, $\mathcal{C}\ell_1 = \mathcal{C}\ell^-_{p,q}$ and the grading automorphism is given by $\widehat{(\ )}$.

4 Spacetime splitting via inner automorphic α-gradings

A vector space endowed with a constant signature (p, q) metric, isomorphic to $\mathbb{R}^{p,q}$, can be identified in a point $x \in M$ as the space T_xM tangent to M in this point, where M is a manifold locally diffeomorphic to the (local) foliation $\mathbb{R} \times \Sigma$ of M. There always exist a 1-form time-like field n, normal to Σ, i.e., $n^2 = g(n, n) > 0$ and $g(n, \vec{\mathbf{v}}) = 0$, for all $\vec{\mathbf{v}} \in T_x\Sigma$ [2]. We denote the metric in $T_xM \simeq \mathbb{R}^{p,q}$ by $g : \sec T_xM \times \sec T_xM \to \mathbb{R}$. The time-like 1-form field $n \in T_x^*M$ can be locally interpreted as being cotangent to the worldline of observer families, i.e., the dual reference frame relative velocity associated with such observers.

The even subalgebra $\mathcal{C}\ell^+_{p,q}$ of $\mathcal{C}\ell_{p,q}$, associated with the graded involution is defined as $\mathcal{C}\ell^+_{p,q} = \{\psi \in \mathcal{C}\ell_{p,q} \,|\, \alpha(\psi) := \hat{\psi} = \psi\}$ and here we define the inner automorphic α-grading $\alpha : \mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{p,q}$ as

$$\alpha(\psi) = n\hat{\psi}n^{-1} \tag{2}$$

From eq.(2) it can be seen that the inner automorphism α is invariant under dilations of n, and without loss of generality it is possible to consider 1-forms n of unitary norm $n^2 = 1$. Eq.(2) is then written as $\alpha(\psi) = n\hat{\psi}n$. Such α-grading is used until Sec. (9).

[1] It can be proved that α is also an algebra isomorphism.

Given $\mathbf{e}^0 \in T_x^*M$, it is well known that n can be obtained via a Lorentz transformation L from the formula $n(x) = L(x)\mathbf{e}^0\tilde{L}(x)$, where L is an element of the group[2] $\mathrm{Spin}_+(1,3)$, in the particular case of $\mathcal{C}\ell_{1,3}$. In the more general case $L \in \mathrm{Spin}_+(p,q)$, a frame $\{\mathbf{e}_p\}$ is said to be adapted to the observer if the following conditions hold:

i) $n = L\mathbf{e}_0\tilde{L}$,

ii) $n^\sharp(\mathbf{e}_p) = 0$,

iii) $\{\mathbf{e}_p\}$ spans the local spacetime related to n.

Note that the automorphisms $\hat{\psi}$ and $\alpha(\psi)$ commute, which can be illustrated by the following diagram:

$$\begin{array}{ccc}
\mathcal{C}\ell_{p,q} & \xrightarrow{\;\;\widehat{}\;\;} & \mathcal{C}\ell_{p,q}^{\pm} \\
\big\downarrow \alpha & & \big\downarrow \alpha \\
\mathcal{C}\ell_{p,q\|}\;\mathcal{C}\ell_{p,q\perp} & \xrightarrow{\;\;\widehat{}\;\;} & \mathcal{C}\ell_{p,q\|}^{\;\pm},\;\mathcal{C}\ell_{p,q\perp}^{\;\pm}
\end{array}$$

Indeed,

$$\alpha(\hat{\psi}) = n\psi n^{-1} = \widehat{n\hat{\psi}n^{-1}} = \widehat{\alpha(\psi)}. \tag{3}$$

Among the vast possibilities to introduce subspaces of $\mathcal{C}\ell_{p,q}$, we define

$$\mathcal{C}\ell_{p,q}^{\|} = \{\psi \in \mathcal{C}\ell_{p,q}\,|\,\alpha(\psi) = n\hat{\psi}n = \psi\}, \qquad \mathcal{C}\ell_{p,q}^{\perp} = \{\psi \in \mathcal{C}\ell_{p,q}\,|\,\alpha(\psi) = n\hat{\psi}n = -\psi\} \tag{4}$$

where $\mathcal{C}\ell_{p,q}^{\|}$ denotes the α-even (spatial) component of $\mathcal{C}\ell_{p,q}$, while $\mathcal{C}\ell_{p,q}^{\perp}$ denotes the α-odd (temporal) component of $\mathcal{C}\ell_{p,q}$, which is shown to be a $\mathbb{Z}_2$-graded algebra with respect to the inner automorphism given by eq.(2). Indeed, given $\psi_\|,\ \phi_\| \in \mathcal{C}\ell_{p,q}^{\|}$ and $\psi_\perp,\ \phi_\perp \in \mathcal{C}\ell_{p,q}^{\perp}$, we have

$$\begin{aligned}
n\widehat{\psi_\|\phi_\|}n &= \psi_\|\phi_\| \in \mathcal{C}\ell_{p,q}^{\|}, \\
n\widehat{\psi_\|\phi_\perp}n &= -\psi_\|\phi_\| \in \mathcal{C}\ell_{p,q}^{\perp}, \\
n\widehat{\psi_\perp\phi_\perp}n &= = \psi_\|\phi_\| \in \mathcal{C}\ell_{p,q}^{\|}
\end{aligned} \tag{5}$$

The graded involution of $\mathcal{C}\ell_{p,q}^{\|}$ is the same of $\mathcal{C}\ell_{p,q}$. The projectors $\pi_\|, \pi_\perp$, defined by relations

$$\pi_\|(\psi) = \frac{1}{2}(\psi + n\hat{\psi}n), \qquad \pi_\perp(\psi) = \frac{1}{2}(\psi - n\hat{\psi}n) \tag{6}$$

can be written as

$$\begin{aligned}
\pi_\|(\psi) &= n\cdot(n\wedge\psi) = (\psi\wedge n)\cdot n = \psi - n\wedge(n\cdot\psi), \\
\pi_\perp(\psi) &= n\wedge(n\cdot\psi) = (\psi\cdot n)\wedge n,
\end{aligned} \tag{7}$$

[2] The group $\mathrm{Spin}_+(p,q)$ is defined as being the group constituted by a product of an even number of unitary norm vectors in $\mathbb{R}^{p,q}$.

from which follows the relations $n \cdot \psi_{\|} = 0, \qquad n \wedge \psi_{\perp} = 0$. Expressions in eq.(7) have a full geometric motivation, since given $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{p,q}$ it follows that

$$\begin{aligned} \mathbf{v} &= \mathbf{v}\mathbf{u}\mathbf{u}^{-1} = (\mathbf{v}\cdot\mathbf{u} + \mathbf{v}\wedge\mathbf{u})\mathbf{u}^{-1} = (\mathbf{v}\cdot\mathbf{u})\mathbf{u}^{-1} + (\mathbf{v}\wedge\mathbf{u})\cdot\mathbf{u}^{-1} \\ &= \mathbf{v}_{\|} + \mathbf{v}_{\perp} \end{aligned} \tag{8}$$

The relations

$$\begin{aligned} &\psi_{\|} = \pi_{\|}(\psi_{\|}), \quad \psi_{\perp} = \pi_{\perp}(\psi_{\perp}), \quad \pi_{\|}\pi_{\perp} = \pi_{\perp}\pi_{\|} = 0, \\ &\pi_{\|}^2 = \pi_{\|}, \quad \pi_{\perp}^2 = \pi_{\perp} \quad \text{and} \quad \pi_{\perp} + \pi_{\|} = 1 \end{aligned} \tag{9}$$

show explicitly that the operators in eq.(7) are indeed projector operators. The convention used asserts that $\psi_{\|}$ [$\psi_{\perp}$] represents the spatial [temporal] component of $\psi \in \mathcal{C}\ell_{p,q}$.

The identities

$$\psi_{\|}n = n\widehat{\psi_{\|}}, \qquad \psi_{\perp}n = -n\widehat{\psi_{\perp}}, \qquad n\psi n = \widehat{\psi_{\|}} - \widehat{\psi_{\perp}} \tag{10}$$

holds and shall be useful in what follows.

From eqs.(5) we see that $\mathcal{C}\ell^{\|}_{p,q}\mathcal{C}\ell^{\|}_{p,q} \hookrightarrow \mathcal{C}\ell^{\|}_{p,q}$, $\mathcal{C}\ell^{\|}_{p,q}\mathcal{C}\ell^{\perp}_{p,q} \hookrightarrow \mathcal{C}\ell^{\perp}_{p,q}$ and $\mathcal{C}\ell^{\perp}_{p,q}\mathcal{C}\ell^{\perp}_{p,q} \hookrightarrow \mathcal{C}\ell^{\|}_{p,q}$. The Clifford product between two multivectors $\psi, \phi \in \mathcal{C}\ell_{p,q}$ is given by

$$\phi\psi = (\phi_{\|} + \phi_{\perp})(\psi_{\|} + \psi_{\perp}) = \phi_{\|}\psi_{\|} + \phi_{\perp}\psi_{\perp} + \phi_{\|}\psi_{\perp} + \phi_{\perp}\psi_{\|} \tag{11}$$

where the parallel and orthogonal components of the Clifford product $\phi\psi$ are respectively given by:

$$(\phi\psi)_{\|} = \phi_{\|}\psi_{\|} + \phi_{\perp}\psi_{\perp} \in \mathcal{C}\ell^{\|}_{p,q}, \qquad (\phi\psi)_{\perp} = \phi_{\|}\psi_{\perp} + \phi_{\perp}\psi_{\|} \in \mathcal{C}\ell^{\perp}_{p,q} \tag{12}$$

Then we have

$$\begin{aligned} (\mathbf{v}\wedge\phi)_{\|} &= \frac{1}{2}(\mathbf{v}\phi + \hat{\phi}\mathbf{v})_{\|} \\ &= \mathbf{v}_{\|}\wedge\phi_{\|} + \mathbf{v}_{\perp}\wedge\phi_{\perp} \end{aligned} \tag{13}$$

But relations $\mathbf{v}_{\perp}\wedge\phi_{\perp} = n\wedge(n\cdot\mathbf{v})\wedge n\wedge(n\cdot\phi_{\perp}) = 0$ follow from $n\wedge n = 0$. Therefore,

$$(\mathbf{v}\wedge\phi)_{\|} = \mathbf{v}_{\|}\wedge\phi_{\|}. \tag{14}$$

Analogously it is immediate to see that

$$(\mathbf{v}\wedge\phi)_{\perp} = \mathbf{v}_{\perp}\wedge\phi_{\|} + \mathbf{v}_{\|}\wedge\phi_{\perp} \tag{15}$$

5 Metric projections

In order to express the metric $g^{\|}_x \equiv g^{\|} : T_x\Sigma \times T_x\Sigma \to \mathbb{R}$ in each point of the spatial manifold Σ, locally given by $g^{\|}(\mathbf{v},\mathbf{u}) = \mathbf{v}_{\|}\cdot\mathbf{u}_{\|}$, the spatial component $g^{\|}$ of the spacetime metric $g : T_xM \times T_xM \to \mathbb{R}$ is obtained from the expression

$$\begin{aligned} \mathbf{v}_{\|}\mathbf{u}_{\|} &= \frac{1}{4}(\mathbf{v} - n\mathbf{v}n)(\mathbf{u} - n\mathbf{u}n) \\ &= \frac{1}{4}(\mathbf{v}\mathbf{u} - \mathbf{v}n\mathbf{u}n - n\mathbf{v}n\mathbf{u} + n\mathbf{v}n^2\mathbf{u}n). \end{aligned} \tag{16}$$

But using $\mathbf{v}n = 2\mathbf{v}\cdot n - n\mathbf{v}$ (similarly for $\mathbf{u}$), it follows that

$$g^{\|}(\mathbf{u},\mathbf{v}) \;=\; g(\mathbf{u},\mathbf{v}) - (\mathbf{v}\cdot n)(\mathbf{u}\cdot n). \tag{17}$$

If a local reference frame in $T_xM \simeq \mathbb{R}^{p,q}$ is adopted, it is possible to express $n = n^\mu \mathbf{e}_\mu, \mathbf{v} = v^\mu \mathbf{e}_\mu, \mathbf{u} = u^\mu \mathbf{e}_\mu$, and from eq.(17) it follows that

$$g^{\|}_{\mu\nu} \;=\; h_{\mu\nu}u^\mu v^\nu, \tag{18}$$

where $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. Then we have

$$\begin{aligned} g^{\|} &= h_{\mu\nu}\mathbf{e}^\mu \otimes \mathbf{e}^\nu = (g_{\mu\nu} - n_\mu n_\nu)\mathbf{e}^\mu \otimes \mathbf{e}^\nu, \\ g^{\perp} &= n_\mu n_\nu \mathbf{e}^\mu \otimes \mathbf{e}^\nu. \end{aligned} \tag{19}$$

6 Decomposition of dual Hodge star operator

We denote pseudoscalars associated with $\mathcal{C}\ell_{p,q}$ and $\mathcal{C}\ell^{\|}_{p,q}$ respectively by $\eta \in \sec \Lambda^{p+q}(T^*_xM)$ and $\tau \in \sec \Lambda^{p+q-1}(T^*_xM)$. From the orientation of $\mathcal{C}\ell_{p,q}$ the orientation in $\mathcal{C}\ell^{\|}_{p,q}$ is defined as

$$\tau := n\cdot\eta \tag{20}$$

or equivalently, $\eta = n\wedge\tau$. Eq.(20) can also be written as

$$\tau = n\eta \tag{21}$$

and therefore relations

$$n\tau = \hat{\tau}n \qquad \text{and} \qquad \hat{\eta} = -n\hat{\tau} = -\tau n \tag{22}$$

immediately follow, since $\tau \in \mathcal{C}\ell^{\|}_{p,q}$.

The dual Hodge star operator (DHSO) is defined in $\mathcal{C}\ell^{\|}_{p,q}$ as

$$\begin{aligned} \star_{\|}\mathbf{v}_{\|} &= \mathbf{v}_{\|}\cdot\tau = \mathbf{v}_{\|}\cdot(n\cdot\eta) \\ &= -n\cdot(\mathbf{v}_{\|}\cdot\eta) \end{aligned} \tag{23}$$

From eqs.(7), expressions $\mathbf{v}_{\|}\cdot\eta = \mathbf{v}\cdot\eta - [n\wedge(n\cdot\mathbf{v})]\cdot\eta$ follows, and also

$$n\cdot(\mathbf{v}_{\|}\cdot\eta) \;=\; n\cdot(\star\mathbf{v}) \tag{24}$$

from which we conclude, using eqs.(23) and (24), that $\star_{\|}\mathbf{v}_{\|} = -n\cdot(\star\mathbf{v})$. We can also write $\star_{\|}\mathbf{v}_{\|} = -n\cdot(\star\mathbf{v})_\perp$ since $n\cdot(\star\mathbf{v})_{\|} = 0$. But $n\wedge(\star\mathbf{v})_\perp = 0$ and then the relation $\star_{\|}\mathbf{v}_{\|} = -n(\star\mathbf{v})_\perp$ implies that

$$(\star\mathbf{v})_\perp = -n(\star_{\|}\mathbf{v}_{\|}) \tag{25}$$

In the particular case of spacetime algebra $\mathcal{C}\ell_{1,3}$ where the α-grading is given by the graded involution, it is well known that $\mathcal{C}\ell^{\|}_{1,3} \simeq \mathcal{C}\ell_{3,0}$, and in the case of the α-grading given by eq.(2) the α-even subalgebra of $\mathcal{C}\ell_{1,3}$ is given by $\mathcal{C}\ell^{\|}_{1,3} \simeq \mathcal{C}\ell_{0,3} \simeq \mathbb{H}\oplus\mathbb{H}$. More details are to be furnished in Sec. (12).

For multivectors $\psi \in \mathcal{C}\ell_{p,q}$ it follows that

$$\star_{\|}\, \psi_{\|} \;=\; n\cdot(\overline{\psi}_{\|}\cdot\eta) \tag{26}$$

Taking again the expression $\psi_{\|} = \psi - n\wedge(n\cdot\psi)$ we can express

$$\overline{\psi}_{\|} \;=\; \star\hat{\psi} - [\widetilde{(n\cdot\tilde{\psi})}\wedge n]\cdot\eta \tag{27}$$

Therefore $n\cdot(\bar{\psi}\cdot\eta) = n\cdot(\star\hat{\psi})$ and we can finally write from eq.(26) that $\star_{\|}\psi_{\|} = n\cdot(\star\hat{\psi})$ and using the properties $n\cdot(\quad)_{\|} = 0$ and $n\wedge(\quad)_{\perp} = 0$ we obtain

$$\star_{\|}\, \psi_{\|} \;=\; n(\star\hat{\psi})_{\perp} \tag{28}$$

from where it follows that $(\star\psi)_{\perp} = n(\star_{\|}\hat{\psi}_{\|})$ and

$$\star_{\|}\, \psi_{\|} \;=\; n\cdot(\star\hat{\psi}) \tag{29}$$

Also,

$$\begin{aligned}\star_{\|}\,(n\cdot\psi) &= \widetilde{(n\cdot\psi)}\cdot\tau \\ &= \widetilde{\psi_{\perp}}\cdot\eta \end{aligned}\tag{30}$$

and therefore

$$\star_{\|}(n\cdot\psi) = \star(\psi_{\perp}) \tag{31}$$

We now prove a similar expression for the parallel component $(\star\psi)_{\|}$ of $\star\psi$. According to relation $\widetilde{\psi_{\perp}} = \frac{1}{2}(\tilde{\psi} - n\bar{\psi}n)$ we can obviously write

$$\widetilde{\psi_{\perp}}\cdot\eta = \frac{1}{2}(\tilde{\psi} - n\bar{\psi}n)\cdot\eta \tag{32}$$

Given $\psi_k \in \sec\Lambda^k(T_xM)$, it follows that

$$\begin{aligned}(n\overline{\psi_k}n)\cdot\eta &= \langle n\overline{\psi_k}n\eta\rangle_{n-k} = -\langle\, n\overline{\psi_k}\hat{\eta}n\rangle_{n-k} \\ &= -n(\widehat{\widetilde{\psi_k}\cdot\eta})n, \end{aligned}\tag{33}$$

and from expression

$$\begin{aligned}\widetilde{\psi_{\perp}}\cdot\eta &= \frac{1}{2}[(\tilde{\psi}\cdot\eta) + n(\widehat{\tilde{\psi}\cdot\eta})n] \\ &= (\tilde{\psi}\cdot\eta)_{\|} = (\star\psi)_{\|} \end{aligned}\tag{34}$$

the relation

$$(\star\psi)_{\|} = \star(\psi_{\perp}) = \star_{\|}(n\cdot\psi) \tag{35}$$

holds.

7 Dirac operator splitting

Consider the Clifford fiber bundle over the manifold M, denoted by $\mathcal{C}\ell(M,g) := \bigcup_{x\in M} \mathcal{C}\ell(T_xM, g_x)$. Given $n \in \sec T_x^*M$, the local decomposition $M = I \times \Sigma$ where I is a interval of $\mathbb{R}$, can be obtained if and only if $n\wedge dn = 0$, from Frobenius theorem. The Dirac operator is denoted by ∂, and the differential $d \equiv \partial\wedge : \sec \Lambda^k(T_x^*M) \to \sec \Lambda^{k+1}(T_x^*M)$ and codifferential $\delta \equiv -\partial\cdot : \sec \Lambda^{k+1}(T_x^*M) \to \sec \Lambda^k(T_x^*M)$ operators are defined in $\mathcal{C}\ell(M,g)$ in terms of the Dirac operator. In what follows we find expression for the components $d_\parallel, d_\perp, \delta_\parallel$ and $\delta_\perp$ and consequently, for $\partial_\parallel, \partial_\perp$. From relations $(\psi\wedge\phi)_\parallel = \psi_\parallel \wedge \phi_\parallel$ we have

$$(\psi\wedge\phi_\parallel)_\parallel = \psi_\parallel\wedge\phi_\parallel \quad \text{and} \quad (\psi\wedge\phi_\perp)_\parallel = 0, \tag{36}$$

and from expression

$$(\psi\wedge\phi)_\perp = \psi_\parallel\wedge\phi_\perp + (\psi_\perp\wedge\phi_\parallel)_\parallel \tag{37}$$

it is immediate that

$$(\psi\wedge\phi_\parallel)_\perp = \psi_\parallel\wedge\phi_\parallel, \qquad (\psi\wedge\phi_\perp)_\perp = \psi_\parallel\wedge\phi_\perp \tag{38}$$

We define the parallel $d_\parallel$ and orthogonal $d_\perp$ components of the differential operator d, from the following expressions

$$d_\parallel\psi_\parallel = (d\psi_\parallel)_\parallel, \quad d_\parallel\psi_\perp = (d\psi_\perp)_\perp, \quad d_\perp\psi_\parallel = (d\psi_\parallel)_\perp, \quad d_\perp\psi_\perp = 0, \tag{39}$$

from where we obtain

$$\begin{aligned} d_\parallel\psi_\parallel &= (d\psi_\parallel)_\parallel \\ &= n\cdot(n\wedge d\psi_\parallel), \end{aligned} \tag{40}$$

$$\begin{aligned} d_\perp\psi_\parallel &= (d\psi_\parallel)_\perp = n\wedge(n\cdot d\psi_\parallel) \\ &= n\wedge \pounds_n\psi_\parallel \end{aligned} \tag{41}$$

and

$$\begin{aligned} d_\perp\psi_\perp = (d\psi_\perp)_\parallel = n\cdot(n\wedge d\psi_\perp) & \\ &= n\cdot\{n\wedge d[n\wedge(n\cdot\psi)]\} \\ &= 0 \end{aligned} \tag{42}$$

Also, the relation

$$\begin{aligned} d_\parallel\psi_\perp = (d\psi_\perp)_\perp &= n\wedge(n\cdot d\psi_\perp) \\ &= n\wedge\{n\cdot[dn\wedge(n\cdot\psi) - n\wedge d(n\cdot\psi)]\} \\ &= -(n\cdot dn)\wedge\psi_\perp - n\wedge d_\parallel(n\cdot\psi) \\ &= n\wedge(\pounds_n n)\wedge(n\cdot\psi) - n\wedge d_\parallel(n\cdot\psi) \end{aligned} \tag{43}$$

is immediately derived. Now, define the multiform field $\Omega_\mu \in \sec \Lambda(T_x^* M)$ as

$$\Omega_\mu = n\partial_\mu n = -(\partial_\mu n)n \tag{44}$$

We can show that $\Omega_\mu \in \sec \Lambda^2(T_x^* M)$, since $n \cdot \partial_\mu n = 0$ and $n^2 = 1$. It is well known that the commutator computed between a multivector and a bivector preserves the grading of the later, and we have

$$\begin{aligned} \frac{1}{2}[\Omega_\mu, \psi_k] &= \langle n\partial_\mu n \psi_k \rangle_k \\ &= -(\partial_\mu n) \cdot (n \wedge \psi_k) - (\partial_\mu n) \wedge (n \cdot \psi_k) \end{aligned} \tag{45}$$

Therefore

$$\begin{aligned} \frac{1}{2}[\Omega_\mu, \psi] &= -(\partial_\mu n) \cdot (n \wedge \psi) - (\partial_\mu n) \wedge (n \cdot \psi) \\ &= n \wedge (\partial_\mu n \cdot \psi) + n \cdot (\partial_\mu n \wedge \psi) \end{aligned} \tag{46}$$

where we used the relation $n \cdot \partial_\mu n = 0$. The commutator between Ω_μ and $\psi_\parallel$ is given by

$$\begin{aligned} \frac{1}{2}[\Omega_\mu, \psi_\parallel] &= -(\partial_\mu n) \cdot (n \wedge \psi_\parallel) - (\partial_\mu n) \wedge (n \cdot \psi_\parallel) \\ &= n \wedge (\partial_\mu n \cdot \psi_\parallel) \\ &= -n \wedge (n \cdot \partial_\mu \psi_\parallel) \end{aligned} \tag{47}$$

where the identity $\partial_\mu n \cdot \psi_\parallel = -n \cdot \partial_\mu \psi_\parallel$ follows from relation $n \cdot \psi_\parallel = 0$. We also get the result

$$\frac{1}{2}[\Omega_\mu, \psi_\parallel] = -(\partial_\mu \psi_\parallel)_\perp \tag{48}$$

since $n \wedge (n \cdot \ \ ) = \pi_\perp(\ \ )$. On the other hand, we have

$$\begin{aligned} \frac{1}{2}[\Omega_\mu, \psi_\perp] &= -(\partial_\mu n) \cdot (n \wedge \psi_\perp) - (\partial_\mu n) \wedge (n \cdot \psi_\perp) \\ &= -(\partial_\mu n) \wedge (n \cdot \psi_\perp) \\ &= -n \cdot (n \wedge \partial_\mu \psi_\perp), \end{aligned} \tag{49}$$

and from the equivalence $n \cdot (n \wedge \ \ ) = \pi_\parallel(\ \ )$ it follows that

$$\frac{1}{2}[\Omega_\mu, \psi_\perp] = -(\partial_\mu \psi_\perp)_\parallel \tag{50}$$

Also

$$\partial_\mu n = -\frac{1}{2}[\Omega_\mu, n] \tag{51}$$

holds, since

$$\frac{1}{2}[\Omega_\mu, n] = \frac{1}{2}[\Omega_\mu, n_\perp] = -(\partial_\mu n_\perp)_\parallel = -(\partial_\mu n)_\parallel = -\partial_\mu n. \tag{52}$$

We used the obvious property that $n = n_\perp$ and, from relation $n \cdot \partial_\mu n = 0$ it follows that

$$(\partial_\mu n)_\perp = 0 \qquad \text{and} \qquad \partial_\mu n = (\partial_\mu n)_\parallel. \tag{53}$$

Alternatively, eq.(51) follows from the definition of Ω_μ. Indeed,

$$\begin{aligned}\frac{1}{2}[\Omega_\mu, n] &= \frac{1}{2}(n\partial_\mu n n - nn\partial_\mu n)\\ &= -\partial_\mu n. \end{aligned} \tag{54}$$

From relations $(\partial_\mu \psi_\parallel)_\perp = -\frac{1}{2}[\Omega_\mu, \psi_\parallel]$ and $(\partial_\mu \psi_\perp)_\parallel = -\frac{1}{2}[\Omega_\mu, \psi_\perp]$ we have

$$\begin{aligned}(\partial_\mu \psi_\parallel)_\parallel &= \partial_\mu \psi_\parallel - (\partial_\mu \psi_\parallel)_\perp\\ &= \partial_\mu \psi_\parallel + \frac{1}{2}[\Omega_\mu, \psi_\parallel]\end{aligned} \tag{55}$$

and

$$\begin{aligned}(\partial_\mu \psi_\perp)_\perp &= \partial_\mu \psi_\perp - (\partial_\mu \psi_\perp)_\parallel\\ &= \partial_\mu \psi_\perp + \frac{1}{2}[\Omega_\mu, \psi_\perp]\end{aligned} \tag{56}$$

8 Covariant and Lie derivative associated with the differential operator

Consider the orthogonal component of the differential operator acting on the parallel component of the multivector $\psi \in \mathcal{C}\ell_{p,q}$:

$$\begin{aligned}(d\psi_\parallel)_\perp &= (\gamma^\mu \wedge \partial_\mu \psi_\parallel)_\perp\\ &= \gamma^\mu_\parallel \wedge (\partial_\mu \psi_\parallel)_\perp + \gamma^\mu_\perp \wedge (\partial_\mu \psi_\parallel)_\parallel\\ &= \gamma^\mu_\parallel \wedge (-\frac{1}{2}[\Omega_\mu, \psi_\parallel]) + n \wedge n^\mu(\partial_\mu \psi_\parallel + \frac{1}{2}[\Omega_\mu, \psi_\parallel])\\ &= \gamma^\mu_\parallel \wedge n \wedge (n \cdot \partial_\mu \psi_\parallel) + n \wedge n^\mu \partial_\mu \psi_\parallel = n \wedge [\gamma^\mu(\partial_\mu n \cdot \psi_\parallel) + n^\mu \partial_\mu \psi_\parallel]\\ &= n \wedge \pounds_n \psi_\parallel\end{aligned} \tag{57}$$

where

$$\begin{aligned}\gamma^\mu_\perp &= n \wedge (n \cdot \gamma^\mu) = nn^\mu,\\ \gamma^\mu_\parallel &= \gamma^\mu - n \wedge (n \cdot \gamma^\mu) = \gamma^\mu - nn^\mu\end{aligned} \tag{58}$$

and $\pounds_n$ is the Lie derivative along n. The expressions above are to be widely used in Sec. (12).

Defining the parallel component of the covariant derivative as

$$D^\parallel_\mu \psi_\parallel = \partial_\mu \psi_\parallel + \frac{1}{2}[\Omega_\mu, \psi_\parallel] \tag{59}$$

we have

$$\begin{aligned}(d\psi_\parallel)_\parallel &= (\gamma^\mu \wedge \partial_\mu \psi_\parallel)_\parallel\\ &= \gamma^\mu_\parallel \wedge (\partial_\mu \psi_\parallel)_\parallel + \gamma^\mu_\perp (\partial_\mu \psi_\parallel)_\perp\\ &= \gamma^\mu_\parallel \wedge (\partial_\mu \psi_\parallel + \frac{1}{2}[\Omega_\mu, \psi_\parallel]) = \gamma^\mu_\parallel \wedge D^\parallel_\mu \psi_\parallel\\ &= d_\parallel \psi_\parallel\end{aligned}$$

Therefore the relation $d_\| = \gamma^\mu_\| \wedge D^\|_\mu$ is immediately obtained.

Now consider the parallel component of the differential operator acting on the orthogonal component of the multivector $\psi \in \mathcal{C}\ell_{p,q}$:

$$\begin{aligned}
(d\psi_\perp)_\| &= (\gamma^\mu \wedge \partial_\mu \psi_\perp)_\| \\
&= \gamma^\mu_\|(-\frac{1}{2}[\Omega_\mu, \psi_\perp]) && (60)\\
&= n \cdot [n \wedge \gamma^\mu \wedge \partial_\mu n \wedge (n \cdot \psi_\perp)], \quad \text{since } n \cdot \partial_\mu n = 0 \text{ and } n \cdot (n \cdot \psi_\perp) = 0 \\
&= n \cdot [n \wedge (\partial_\mu \wedge n) \wedge (n \cdot \psi_\perp)] \\
&= 0. && (61)
\end{aligned}$$

Finally, for the orthogonal component of $d\psi_\perp$ we have

$$\begin{aligned}
(d\psi_\perp)_\perp &= (\gamma^\mu \wedge \partial_\mu \psi_\perp)_\perp \\
&= \gamma^\mu_\| \wedge (\partial_\mu \psi_\perp)_\perp + \gamma^\mu_\perp (\partial_\mu \psi_\perp)_\| \\
&= \gamma^\mu_\| \wedge (\partial_\mu \psi_\perp + \frac{1}{2}[\Omega_\mu, \psi_\perp]) + \gamma^\mu_\perp \wedge (-\frac{1}{2}[\Omega_\mu, \psi_\perp]) \\
&= \gamma^\mu_\|(D_\mu \psi_\perp) + n \wedge \pounds_n n \wedge (n \cdot \psi_\perp)
\end{aligned}$$

But it is well known that $\gamma^\mu_\| \wedge D_\mu \psi_\perp = \gamma^\mu_\| \wedge (\partial_\mu \psi_\perp) + \gamma^\mu_\| \wedge \frac{1}{2}[\Omega_\mu, \psi_\perp]$ and, from eqs.(60) and (61) it follows that, $\gamma^\mu_\|(-\frac{1}{2}[\Omega_\mu, \psi_\perp]) = 0$. Therefore

$$\begin{aligned}
\gamma^\mu_\| \wedge \partial_\mu \psi_\perp &= \gamma^\mu_\| \wedge \{\partial_\mu [n \wedge (n \cdot \psi)]\} \\
&= \gamma^\mu_\| \wedge [\partial_\mu n \wedge (n \cdot \psi) + n \wedge \partial_\mu (n \cdot \psi)] \\
&= -n \wedge \gamma^\mu_\| \wedge \partial_\mu (n \cdot \psi) && (62)
\end{aligned}$$

But

$$\begin{aligned}
\frac{1}{2}[\Omega_\mu, n \cdot \psi] &= -n \wedge (\partial_\mu n \cdot (n \cdot \psi)) + n \cdot (\partial_\mu n \wedge (n \cdot \psi)) \\
&= -n \wedge (\partial_\mu n \cdot (n \cdot \psi)) - (n \cdot \partial_\mu n)(n \cdot \psi) + \partial_\mu n \wedge [n \cdot (n \cdot \psi)] \\
&= -n \wedge (\partial_\mu n \cdot (n \cdot \psi)) + \partial_\mu n \wedge [n \wedge (n \cdot \psi)] && (63)
\end{aligned}$$

and then $n \wedge \frac{1}{2}[\Omega_\mu, n \cdot \psi] = 0$. Therefore it follows that

$$\begin{aligned}
\gamma^\mu_\| \wedge \partial_\mu \psi_\perp &= -n \wedge \gamma^\mu_\| \wedge \partial_\mu (n \cdot \psi) \\
&= -n \wedge \gamma^\mu_\| \wedge D_\mu (n \cdot \psi) && (64)
\end{aligned}$$

Finally we have

$$(d\psi_\perp)_\perp = -n \wedge \gamma^\mu_\| \wedge D_\mu (n \cdot \psi) + n \wedge \pounds_n n \wedge (n \cdot \psi_\perp) \tag{65}$$

Using the definition of the operator $\pi_\perp$, we have

$$\pounds_n[\pi_\perp(\psi)] = \pounds_n n \wedge (n \cdot \psi) + n \wedge \pounds_n (n \cdot \psi) \tag{66}$$

On the other hand we know that

$$\begin{aligned}
\pounds_n(n\cdot\psi) &= n\cdot d(n\cdot\psi) + d[n\cdot(n\cdot\psi)] \\
&= \pounds_n n\wedge(n\cdot\psi) + n\wedge[n\cdot d(n\cdot\psi)] \\
&= \pounds_n n\wedge(n\cdot\psi) + n\wedge\{n\cdot[d(n\cdot\psi)+n\cdot(d\psi)]\},\ \text{since } n\cdot(n\cdot d\psi)=0 \\
&= \pounds_n n\wedge(n\cdot\psi) + \pi_\perp(\pounds_n\psi)
\end{aligned}$$

and therefore

$$\begin{aligned}
\pounds_n[\pi_\perp(\psi)] &= \pounds_n n\wedge(n\cdot\psi) + n\wedge\pounds_n(n\cdot\psi) \\
&= \pounds_n n\wedge(n\cdot\psi) + n\wedge\pounds_n n\wedge(n\cdot\psi) + n\wedge\pi_\perp(\pounds_n\psi)
\end{aligned} \tag{67}$$

8.1 Necessary condition for $[\pounds_n, d_\|]=0$

Since the Lie derivative $\psi\in\mathcal{C}\ell_{p,q}$ calculated along n is given by

$$\begin{aligned}
\pounds_n d\psi &= d(n\cdot d\psi) + n\cdot d(d\psi) \\
&= d(n\cdot d\psi) + d[d(n\cdot\psi)] \\
&= d\pounds_n\psi
\end{aligned} \tag{68}$$

and then $d\pounds_n = \pounds_n d$, it follows that

$$\begin{aligned}
\pounds_n d_\|\psi &= \pounds_n[n\cdot(n\wedge d\psi)] \\
&= d(\pounds_n\psi) - \pounds_n n\wedge(n\cdot d\psi) - n\wedge\pounds_n(n\cdot d\psi) \\
&= d(\pounds_n\psi) - n\wedge\{d[n\cdot(n\cdot d\psi)]\} - n\wedge\{n\cdot d[d(n\cdot\psi)]\} - \pounds_n n\wedge(n\cdot d\psi) \\
&= d_\|(\pounds_n\psi) - \pounds_n n\wedge(n\cdot d\psi)
\end{aligned}$$

Therefore

$$\pounds_n(d_\|\psi) = d_\|(\pounds_n\psi) - \pounds_n n\wedge(n\cdot d\psi) \tag{69}$$

showing that $\pounds_n$ and $d_\|$ commute if and only if $\pounds_n n = 0$, or equivalently when the acceleration $\nabla_n n$ is identically null.

9 Covariant and Lie derivatives related to codifferential operator

Consider initially the parallel component $\delta_\|$ of the codifferential operator δ. We define

$$\begin{aligned}
\delta_\|\psi_\| &= (\delta\psi_\|)_\| \\
&= -(\partial\cdot\psi_\|)_\|
\end{aligned} \tag{70}$$

From the definition above we see that

$$\begin{aligned}
(\partial\cdot\psi_\|)_\| &= (\gamma^\mu\cdot\partial_\mu\psi_\|)_\| \\
&= (\gamma^\mu_\|\cdot\partial_\mu\psi_\|)_\| + (\gamma^\mu_\perp\cdot\partial_\mu\psi_\|)_\perp \\
&= \gamma^\mu_\|\cdot(D^\|_\mu\psi_\|) - n\cdot\left(\frac{1}{2}[\Omega(n),\psi_\|]\right),
\end{aligned} \tag{71}$$

where $\Omega(n) := n^\mu \Omega_\mu$ and the covariant derivative is given by $D^{\|}_\mu \psi_\| = \partial_\mu \psi_\| + \frac{1}{2}[\Omega_\mu, \psi_\|]$. On the other hand,

$$
\begin{aligned}
\partial_\| \wedge (\star_\| \widehat{\psi}_\|) &= \partial_\| \wedge [n(\star \widehat{\widehat{\psi}}_\perp)] \\
&= d_\| [n(\star\psi)_\perp] = d_\| \wedge (n\widetilde{\psi}_\| \eta) \\
&= \gamma^\mu_\| \wedge \left(\partial_\mu (n\tilde{\psi}_\| \eta) + \frac{1}{2}[\Omega_\mu, n\tilde{\psi}_\| \eta] \right) \qquad (72)
\end{aligned}
$$

But from relations

$$
\begin{aligned}
[\Omega_\mu, n\widetilde{\psi}_\| \eta] &= (\Omega_\mu n \widetilde{\psi}_\| \eta - n\widetilde{\psi}_\| \eta \Omega_\mu) \\
&= [\Omega_\mu, n]\widetilde{\psi}_\| \eta + n[\Omega_\mu, \widetilde{\psi}_\|]\eta \qquad (73)
\end{aligned}
$$

we obtain, using $\partial_\mu \eta = 0$, the expressions

$$
\begin{aligned}
\partial_\| \wedge (\star_\| \widehat{\psi_\|}) &= \gamma^\mu_\| \wedge \left(\partial_\mu n)\widetilde{\psi_\|}\eta + n(\partial_\mu \widetilde{\psi_\|})\eta + \frac{1}{2}[\Omega_\mu, n]\widetilde{\psi_\|}\eta + \frac{1}{2} n[\Omega_\mu, \widetilde{\psi_\|}]\eta \right] \\
&= \gamma^\mu_\| \wedge [n(\partial_\mu \widetilde{\psi_\|})\eta + \frac{1}{2} n[\Omega_\mu, \widetilde{\psi_\|}]\eta], \quad \text{since } \partial_\mu n = -\frac{1}{2}[\Omega_\mu, n] \\
&= \gamma^\mu_\| \wedge [n(D^{\|}_\mu \widetilde{\psi_\|})\eta] \\
&= -\frac{n}{2}\left[\gamma^\mu_\| \partial_\mu \widetilde{\psi_\|} - (\widehat{\partial_\mu \widetilde{\psi_\|}})\gamma^\mu_\| + \left(\gamma^\mu_\| \frac{1}{2}[\Omega_\mu, \widetilde{\psi_\|}] - \frac{1}{2}\widehat{[\Omega_\mu, \widetilde{\psi_\|}]}\gamma^\mu_\| \right) \eta \right] \\
&= -n[\gamma^\mu_\| \cdot (D^{\|}_\mu \widetilde{\psi_\|})]\eta \qquad (74)
\end{aligned}
$$

Now the DHSO on the expression above is calculated:

$$
\begin{aligned}
\star_\|^{-1} [\partial_\| \wedge (\star_\| \widehat{\psi_\|})] &= \tilde{\tau}^{-1}\{-[(\widetilde{\gamma^\mu_\| \cdot D^{\|}_\mu \widetilde{\psi_\|}})n]\eta\} \\
&= -\tilde{\tau}^{-1}\tilde{\eta}[\widetilde{\gamma^\mu_\| \cdot (D^{\|}_\mu \widetilde{\psi_\|})}]n = -n[(D^{\|}_\mu \psi_\|) \cdot \gamma^\mu_\|]n \\
&= -\gamma^\mu_\| \cdot D^{\|}_\mu \psi_\| \qquad (75)
\end{aligned}
$$

Comparing this last expression with eq.(71), we obtain

$$
(\partial \cdot \psi_\|)_\| = -\star_\|^{-1} d_\| \star_\| \widehat{\psi_\|} - n \cdot \left(\frac{1}{2}[\Omega(n), \psi_\|] \right) \qquad (76)
$$

But as

$$
\begin{aligned}
n \cdot \frac{1}{2}[\Omega(n), \psi_\|] &= n^\mu n \cdot \frac{1}{2}[\Omega_\mu, \psi_\|] \\
&= -n^\mu \{n \cdot [(\partial_\mu n) \cdot (n \wedge \psi_\|)]\} \\
&= n^\mu \{(\partial_\mu n) \cdot [n \cdot (n \wedge \psi_\|)]\} \\
&= \pounds_n n \cdot \psi_\| \qquad (77)
\end{aligned}
$$

it follows that

$$
(\partial \cdot \psi_\|)_\| = -\star_\|^{-1} d_\| \star_\| \widehat{\psi_\|} - (\pounds_n n) \cdot \psi_\| \qquad (78)
$$

10 Equivalent dual decompositions

We have defined from eq.(2) the α-grading given by

$$\alpha(\psi) = n\hat{\psi}n^{-1} \tag{79}$$

From eqs.(20, 21), as $\eta \in \sec \Lambda^{p+q}(T_x^*M)$ and therefore $\eta^2 = \pm 1$, it can be seen that

$$\begin{aligned}
\alpha(\psi) &= n\hat{\psi}n^{-1} \\
&= n\eta^2\hat{\psi}\eta^{-2}n^{-1} \\
&= n\eta\psi\eta\eta^{-1}\eta^{-1}n^{-1} \\
&= (-1)^{|\psi|(p+q-1)}\,\tau\psi\tau^{-1}
\end{aligned} \tag{80}$$

Therefore there is an equivalent decomposition of elements in $\mathcal{C}\ell_{p,q}$:

1. From the 1-form field $n \in \sec T_x^*M$ in the normal bundle given by

$$\alpha(\psi)_n := n\hat{\psi}n^{-1} \tag{81}$$

or

2. Via the field defined by the volume element $\tau \in \sec \Lambda^{p+q-1}(T_x^*M)$ associated with $\mathcal{C}\ell_{p,q}^{\|}$, where the decomposition (given by eq.(81)) is now rewritten as

$$\alpha(\psi)_\tau := (-1)^{|\psi|(p+q-1)}\,\tau\psi\tau^{-1} \tag{82}$$

from eq.(80).

As decompositions given by $\alpha(\psi)_n$ and $\alpha(\psi)_\tau$ are identical, it is immediate that the action of differential operators on such automorphisms are also equivalent. Indeed we explicitly calculate below the action of $d : \sec \Lambda^p(T_x^*M) \to \sec \Lambda^{p+1}(T_x^*M)$ in each one of the decompositions given by eqs.(79) and (80):

$$d\alpha(\psi)_n = dn\hat{\psi}n^{-1} - nd\hat{\psi}n^{-1} + (-1)^{|\psi|}n\hat{\psi}dn^{-1} \tag{83}$$

$$d\alpha(\psi)_\tau = (-1)^{|\psi|(p+q-1)}\,(dn\hat{\psi}n^{-1} - nd\hat{\psi}n^{-1} - (-1)^{|\psi|}n\hat{\psi}dn^{-1}) \tag{84}$$

Now, the action of the codifferential operator $\delta : \sec \Lambda^p(T_x^*M) \to \sec \Lambda^{p-1}(T_x^*M)$ in each one of the decompositions, considering $\delta\phi = \star^{-1}d\star\hat{\phi}$ and $\star\phi = \tilde{\phi}\eta$ ($\phi \in \mathcal{C}\ell_{p,q}$), is given respectively by

$$\begin{aligned}
\delta\alpha(\psi)_n &= \star^{-1}d\star\widehat{\alpha(\psi)_n} \\
&= \star^{-1}d\,(n\tilde{\psi}n^{-1}\eta) \\
&= \star^{-1}\{dn^{-1}\,\tilde{\psi}n\eta - n^{-1}\,d\tilde{\psi}\,n\eta - (-1)^{|\psi|}n^{-1}\tilde{\psi}\,dn\eta\} \\
&= \beta\{\eta n\psi(-dn^{-1})\eta - \eta n(-\widehat{d\psi})n^{-1}\eta - (-1)^{|\psi|}\eta(-dn)\psi n^{-1}\eta\} \\
&= -(-1)^{(p+q-1)}\,\beta\{n\hat{\psi}dn^{-1} + nd\psi n^{-1} + (-1)^{|\psi|}dn\hat{\psi}n^{-1}\},
\end{aligned} \tag{85}$$

where the constant $\beta := (-1)^{q+|\psi|(p+q-|\psi|)}$ arises from the property $\star^{-1} = \beta\star$, and by

$$\begin{aligned}
\delta\alpha(\psi)_\tau &= \star^{-1} d \star \widehat{\alpha(\psi)_\tau} \\
&= \star^{-1} d\,(n\eta\bar{\psi}\eta^{-1}n^{-1}\eta) \\
&= \star^{-1}\{dn^{-1}\eta^{-1}\,\bar{\psi}n - n^{-1}\eta^{-1}\,d\bar{\psi}\,n - (-1)^{|\psi|}n^{-1}\eta^{-1}\bar{\psi}\,dn\} \\
&= \beta\{n\hat{\psi}\eta^{-1}(-dn^{-1})\eta - n(\widetilde{-d\bar{\psi}})\eta^{-1}n^{-1}\eta - (-1)^{|\psi|}\eta(-dn)\bar{\psi}\eta^{-1}n^{-1}\eta\} \\
&= (-1)^{|\psi|(p+q-1)}\,\beta\{-n\hat{\psi}dn^{-1} + nd\psi n^{-1} - (-1)^{|\psi|}dn\hat{\psi}n^{-1}\}.
\end{aligned} \tag{86}$$

Since the Dirac operator ∂ is given by $d-\delta$, it follows from eqs.(83, 84, 85, 86), that $\partial(\alpha(\psi)_n) = \partial(\alpha(\psi)_\tau)$ can be written as

$$\partial(\alpha(\psi)_n) = (1+(-1)^{|\psi|}\beta\Delta)dn\,\hat{\psi}\,n^{-1} - n(d\hat{\psi} - \beta\Delta\,d\psi)n^{-1} + (\beta\Delta - (-1)^{|\psi|})n\hat{\psi}\,dn^{-1} \tag{87}$$

where $\Delta := (-1)^{|\psi|(p+q-1)}$. The expression above can be straightforwardly written in the particular case of Minkowski spacetime as

$$\partial(\alpha(\psi)_n) = \partial(\alpha(\psi)_\tau) = -2\left((-1)^{|\psi|}nd\psi n^{-1} + n\psi dn^{-1}\right) \tag{88}$$

11 k-vectorial inner automorphisms

Heretofore we treated inner automorphisms of $\mathcal{C}\ell_{p,q}$ given by

$$\alpha(\psi)_n := n\hat{\psi}n^{-1} \tag{89}$$

where $n \in T_x^*M$ denotes a 1-form field. We can generalize these automorphisms, considering more general automorphisms of $\mathcal{C}\ell_{p,q} \simeq \sec\mathcal{C}\ell(M,g)$ generated by k-form fields instead of 1-form fields, from definition

$$\alpha^{(k)}(\psi) := \phi_{(k)}\psi\phi_{(k)}^{-1} \tag{90}$$

where $\phi_{(k)} \in \sec\Lambda^k(T_x^*M)$. The projector operators given by eqs.(6) can be generalized by expressions

$$\pi_{\|}^{(k)}(\psi) = \frac{1}{2}(\psi + \phi_{(k)}\psi\phi_{(k)}^{-1}), \qquad \pi_{\perp}^{(k)}(\psi) = \frac{1}{2}(\psi - \phi_{(k)}\psi\phi_{(k)}^{-1}) \tag{91}$$

In the subsequent subsections we explicitly express α-gradings for all elements of $\mathcal{C}\ell_{1,3}$. Consider $\phi^{(k)} \neq \psi$, otherwise we have the trivial case $\alpha^{(k)}(\psi) = \psi$. In what follows consider $1 \leq i,j,k \leq 3$ distinct indices.

11.1 Bivectorial inner automorphisms

In this case the decomposition is defined by

$$\alpha^{(2)}(\psi) := \phi_{(2)}\psi\phi_{(2)}^{-1} \tag{92}$$

where $\phi_{(2)} \in \sec\Lambda^2(T_x^*M)$ is a 2-form field. We have the following cases $(i,j,k = 1,2,3)$:

a) $\psi \in \Lambda^1(T_x^*M)$:

ψ	$\phi_{(2)}$	$\alpha^{(2)}(\psi)$	$\pi_{\parallel}^{(2)}(\psi)$	$\pi_{\perp}^{(2)}(\psi)$
$\mathbf{e}^0$	$\mathbf{e}^{0j}$	$-\mathbf{e}^0$	0	$\mathbf{e}^0$
$\mathbf{e}^0$	$\mathbf{e}^{ij}$	$\mathbf{e}^0$	$\mathbf{e}^0$	0
$\mathbf{e}^i$	$\mathbf{e}^{0j}$	$\mathbf{e}^i$	$\mathbf{e}^i$	0
$\mathbf{e}^i$	$\mathbf{e}^{ik}$	$-\mathbf{e}^i$	0	$\mathbf{e}^i$
$\mathbf{e}^i$	$\mathbf{e}^{0i}$	$-\mathbf{e}^i$	0	$\mathbf{e}^i$
$\mathbf{e}^i$	$\mathbf{e}^{jk}$	$\mathbf{e}^i$	$\mathbf{e}^i$	0

b) $\psi \in \Lambda^2(T_x^*M)$:

ψ	$\phi_{(2)}$	$\alpha^{(2)}(\psi)$	$\pi_{\parallel}^{(2)}(\psi)$	$\pi_{\perp}^{(2)}(\psi)$
$\mathbf{e}^{0i}$	$\mathbf{e}^{0j}$	$-\mathbf{e}^{0i}$	0	$\mathbf{e}^{0i}$
$\mathbf{e}^{0i}$	$\mathbf{e}^{ij}$	$-\mathbf{e}^{0i}$	0	$\mathbf{e}^{0i}$
$\mathbf{e}^{0i}$	$\mathbf{e}^{jk}$	$\mathbf{e}^{0i}$	$\mathbf{e}^{0i}$	0
$\mathbf{e}^{ij}$	$\mathbf{e}^{0i}$	$-\mathbf{e}^{ij}$	0	$\mathbf{e}^{ij}$
$\mathbf{e}^{ij}$	$\mathbf{e}^{0k}$	$\mathbf{e}^{ij}$	$\mathbf{e}^{ij}$	0
$\mathbf{e}^{ij}$	$\mathbf{e}^{ik}$	$-\mathbf{e}^{ij}$	0	$\mathbf{e}^{ij}$

c) $\psi \in \Lambda^3(T_x^*M)$:

ψ	$\phi_{(2)}$	$\alpha^{(2)}(\psi)$	$\pi_{\parallel}^{(2)}(\psi)$	$\pi_{\perp}^{(2)}(\psi)$
$\mathbf{e}^{0ij}$	$\mathbf{e}^{0j}$	$\mathbf{e}^{0ij}$	$\mathbf{e}^{0ij}$	0
$\mathbf{e}^{0ij}$	$\mathbf{e}^{ij}$	$\mathbf{e}^{0i}$	$\mathbf{e}^{0ij}$	0
$\mathbf{e}^{0ij}$	$\mathbf{e}^{jk}$	$-\mathbf{e}^{0ij}$	0	$\mathbf{e}^{0ij}$
$\mathbf{e}^{123}$	$\mathbf{e}^{0i}$	$-\mathbf{e}^{123}$	0	$\mathbf{e}^{123}$
$\mathbf{e}^{123}$	$\mathbf{e}^{ij}$	$\mathbf{e}^{123}$	$\mathbf{e}^{123}$	0

d) $\psi \in \Lambda^4(T_x^*M)$:

ψ	$\phi_{(2)}$	$\alpha^{(2)}(\psi)$	$\pi_{\parallel}^{(2)}(\psi)$	$\pi_{\perp}^{(2)}(\psi)$
$\mathbf{e}^{0123}$	$\mathbf{e}^{\mu\nu}$	$\mathbf{e}^{0123}$	$\mathbf{e}^{0123}$	0

11.2 Trivectorial inner automorphisms

Now the α-grading is defined by

$$\alpha^{(3)}(\psi) := \phi_{(3)}\psi\phi_{(3)}^{-1} \tag{93}$$

where $\phi_{(3)} \in \sec \Lambda^3(T_x^*M)$ is a 3-form field.

a) $\psi \in \Lambda^1(T_x^* M)$:

ψ	$\phi_{(3)}$	$\alpha^{(3)}(\psi)$	$\pi_{\parallel}^{(3)}(\psi)$	$\pi_{\perp}^{(3)}(\psi)$
$\mathbf{e}^0$	$\mathbf{e}^{0ij}$	$\mathbf{e}^0$	$\mathbf{e}^0$	0
$\mathbf{e}^0$	$\mathbf{e}^{123}$	$-\mathbf{e}^0$	0	$\mathbf{e}^0$
$\mathbf{e}^i$	$\mathbf{e}^{0ij}$	$\mathbf{e}^i$	$\mathbf{e}^i$	0
$\mathbf{e}^i$	$\mathbf{e}^{0jk}$	$-\mathbf{e}^i$	0	$\mathbf{e}^i$
$\mathbf{e}^i$	$\mathbf{e}^{123}$	$\mathbf{e}^i$	$\mathbf{e}^i$	0

b) $\psi \in \Lambda^2(T_x^* M)$:

ψ	$\phi_{(3)}$	$\alpha^{(3)}(\psi)$	$\pi_{\parallel}^{(3)}(\psi)$	$\pi_{\perp}^{(3)}(\psi)$
$\mathbf{e}^{0i}$	$\mathbf{e}^{0ij}$	$\mathbf{e}^{0i}$	$\mathbf{e}^{0i}$	0
$\mathbf{e}^{0i}$	$\mathbf{e}^{123}$	$-\mathbf{e}^{0i}$	0	$\mathbf{e}^{0i}$
$\mathbf{e}^{ij}$	$\mathbf{e}^{0ij}$	$\mathbf{e}^{ij}$	$\mathbf{e}^{ij}$	0
$\mathbf{e}^{ij}$	$\mathbf{e}^{0ik}$	$-\mathbf{e}^{ij}$	0	$\mathbf{e}^{ij}$
$\mathbf{e}^{ij}$	$\mathbf{e}^{123}$	$\mathbf{e}^{ij}$	$\mathbf{e}^{ij}$	0

c) $\psi \in \Lambda^3(T_x^* M)$:

ψ	$\phi_{(3)}$	$\alpha^{(3)}(\psi)$	$\pi_{\parallel}^{(3)}(\psi)$	$\pi_{\perp}^{(3)}(\psi)$
$\mathbf{e}^{0ij}$	$\mathbf{e}^{0ik}$	$\mathbf{e}^{0ij}$	$\mathbf{e}^{0ij}$	0
$\mathbf{e}^{0ij}$	$\mathbf{e}^{123}$	$-\mathbf{e}^{0ij}$	0	$\mathbf{e}^{0ij}$
$\mathbf{e}^{123}$	$\mathbf{e}^{0ij}$	$-\mathbf{e}^{123}$	0	$-\mathbf{e}^{123}$

d) $\psi \in \Lambda^4(T_x^* M)$:

ψ	$\phi_{(3)}$	$\alpha^{(3)}(\psi)$	$\pi_{\parallel}^{(3)}(\psi)$	$\pi_{\perp}^{(3)}(\psi)$
$\mathbf{e}^{0123}$	$\mathbf{e}^{\mu\nu\sigma}$	$-\mathbf{e}^{0123}$	0	$\mathbf{e}^{0123}$

11.3 Tetravectorial inner automorphisms

In this last case concerning k-vetorial decompositions in $\mathcal{C}\ell_{1,3}$, the α-grading is defined by

$$\alpha^{(4)}(\psi) := \mathbf{e}_{0123}\psi\mathbf{e}_{0123}{}^{-1} \tag{94}$$

where $\mathbf{e}_{0123} \in \sec \Lambda^4(T_x M)$ denotes a volume element field in $\mathbb{R}^{1,3} \simeq T_x M$. All cases can be shown in the following table, for $\psi \in \Lambda^i(T_x^* M), i = 1, 2, 3$:

ψ	$\alpha^{(4)}(\psi)$	$\pi_{\parallel}^{(4)}(\psi)$	$\pi_{\perp}^{(4)}(\psi)$
$\mathbf{e}^{\mu}$	$-\mathbf{e}^{\mu}$	0	$\mathbf{e}^{\mu}$
$\mathbf{e}^{\mu\nu}$	$\mathbf{e}^{\mu\nu}$	$\mathbf{e}^{\mu\nu}$	0
$\mathbf{e}^{\mu\nu\sigma}$	$-\mathbf{e}^{\mu\nu\sigma}$	0	$\mathbf{e}^{\mu\nu\sigma}$

This is clearly the case given by the grade involution $\alpha^{(4)}(\psi) = \hat{\psi}$.

11.4 Multivectorial inner automorphisms

This case is simply the extension by linearity of the inner automorphism defined in Subsecs. (11.1 - 11.3), as

$$\alpha^{\phi}(\psi) := \sum_{b=1}^{4} \alpha^{(b)}(\psi) \tag{95}$$

where $\alpha^{(1)}(\psi) \equiv \alpha(\psi)_n = n\hat{\psi}n^{-1}$. We can also express

$$\alpha^{\phi}(\psi) := \phi\psi\phi^{-1}, \qquad \phi, \psi \in \mathcal{C}\ell_{p,q}. \tag{96}$$

12 Dirac-Hestenes equation decomposition

Hestenes approach to Dirac theory [3, 4] is based on the real Clifford algebra $\mathcal{C}\ell_{1,3}$, instead of the Dirac algebra $\mathbb{C}\otimes\mathcal{C}\ell_{1,3} \simeq \mathcal{M}(4,\mathbb{C})$ used to formulate Dirac theory [5, 6]. In such approach Hestenes asserts that a spinor field ψ is an element of the even subalgebra $\mathcal{C}\ell_{1,3}^{+}$.

Dirac equation can be written as [5]

$$i\gamma_\mu\partial^\mu\psi = m\psi \tag{97}$$

where $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^\dagger$ is a column-vector in $\mathbb{C}^4$. In order to represent vectors, spinors and operators in a unique formalism, the Clifford algebra, column-spinors are substituted by square matrices [3, 4, 9]. Using the standard representation of Dirac matrices, the idempotent

$$f = \frac{1}{2}(1+\mathbf{e}_0)(1+i\mathbf{e}_1\mathbf{e}_2) \simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{98}$$

is chosen in such a way that the spinor ψ can be expressed as an algebraic spinor [7, 8].

$$\mathbb{C}^4 \ni \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \simeq \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}(4,\mathbb{C})f \simeq (\mathbb{C}\otimes\mathcal{C}\ell_{1,3})f$$

Since $i\psi = \psi\mathbf{e}_1\mathbf{e}_2$, the spinor Φ is defined in $\mathcal{C}\ell_{1,3}\frac{1}{2}(1+\mathbf{e}_0)$.

The automorphism $\alpha(\psi) = \mathbf{e}_0\hat{\psi}\mathbf{e}_0{}^{-1}$, given by eq.(2) in the particular case where $n = S\mathbf{e}_0S^{-1} = \mathbf{e}_0$, induces a $\mathbb{Z}_2$-grading in $\mathcal{C}\ell_{1,3}$, given by [11, 12]:

$$\mathcal{C}\ell_{1,3}^{\|} = \text{span}\,\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{123}\} \tag{99}$$

$$\mathcal{C}\ell_{1,3}^{\perp} = \text{span}\,\{\mathbf{e}_0, \mathbf{e}_{01}, \mathbf{e}_{02}, \mathbf{e}_{03}, \mathbf{e}_{012}, \mathbf{e}_{023}, \mathbf{e}_{031}, \mathbf{e}_{0123}\} \tag{100}$$

The subalgebra $\mathcal{C}\ell_{1,3}^{+}$ can be mapped in $\mathcal{C}\ell_{1,3}^{-}$ if elements of $\mathcal{C}\ell_{1,3}^{+}$ are multiplied by $\mathbf{e}_0$. Indeed Φ can be written as [9]

$$\Phi = \Phi_0 + \Phi_1 = (\Phi_0 + \Phi_1)\frac{1}{2}(1+\mathbf{e}_0) = \frac{1}{2}(\Phi_0 + \Phi_1\mathbf{e}_0) + \frac{1}{2}(\Phi_1 + \Phi_0\mathbf{e}_0).$$

We immediately verify that $\Phi_0 = \Phi_1\mathbf{e}_0$ e $\Phi_1 = \Phi_0\mathbf{e}_0$. Taking the real part of eq.(97) we obtain

$$\mathbf{e}_\mu\partial^\mu\Phi\mathbf{e}_2\mathbf{e}_1 = m\Phi, \quad \Phi \in \mathcal{C}\ell_{1,3}\frac{1}{2}(1+\mathbf{e}_0).$$

The even component (associated with the graded involution) of the equation above with respect to the graded involution is the Dirac-Hestenes equation [4, 9]:

$$\partial\psi\mathbf{e}_1\mathbf{e}_2 - m\psi\mathbf{e}_0 = 0, \quad \psi \in \mathcal{C}\ell_{1,3}^+. \tag{101}$$

The spinor field $\psi : \mathbb{R}^{1,3} \to \mathcal{C}\ell_{1,3}^+$ is denominated Dirac-Hestenes spinor field (DHSF) which is an operatorial spinor, since it generates the observables in Dirac theory. A DHSF admits the canonical decomposition [9] $\psi = \sqrt{\rho}\exp(\mathbf{e}_5\beta/2)R$, denoting $\mathbf{e}_5 = i\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, if $\psi\tilde{\psi} \neq 0$, where $\sqrt{\rho}$ denotes a dilation, $\exp(\mathbf{e}_5\beta/2)$ is a dual rotation, β denotes the Yvon-Takabayasi angle [13, 14] and the bivector R, element of the group Spin$_+$(1,3) $\simeq$ SL(2,$\mathbb{C}$), is a Lorentz operator.

Now, taking $n = S\mathbf{e}^0S^{-1}$ e $\sigma = S\mathbf{e}^{12}S^{-1}$, where S is a unitary operator, it is immediate that

$$n^2 = 1, \qquad \sigma^2 = -1, \qquad [n,\sigma] = 0, \tag{102}$$

and from the idempotent given by eq.(98) the idempotent $\frac{1}{2}(1+n)\frac{1}{2}(1+\sigma)$ can be constructed. On the other hand, under an arbitrary α-grading it is well known that it is always possible to led Dirac-Hestenes equation (eq.(101)) to the more general form given by [11]

$$\breve{\partial}\psi\sigma + m\psi n = 0 \qquad \psi \in \mathcal{C}\ell^{\|} \tag{103}$$

where $\breve{\partial}\psi = \pi_{\|}(\partial\psi)n + \pi_\perp(\partial\psi)$ and n is clearly α-odd, while σ is α-even.

Considering the automorphism $\alpha(\psi) = n\hat{\psi}n^{-1}$, it follows that given a spatial reference frame $\{\mathfrak{n}^1, \mathfrak{n}^2, \mathfrak{n}^3\} \in T_x^*\Sigma \simeq \mathbb{R}^{0,3}$ adapted to $n := \mathfrak{n}^0 = S\mathbf{e}^0S^{-1}$, such automorphism induces a $\mathbb{Z}_2$-grading in $\mathcal{C}\ell_{1,3}$, given by

$$\begin{aligned}
\mathcal{C}\ell_{1,3}^{\|} &= \text{span}\,\{1, \mathfrak{n}^1, \mathfrak{n}^2, \mathfrak{n}^3, \mathfrak{n}^{12}, \mathfrak{n}^{23}, \mathfrak{n}^{31}, \mathfrak{n}^{123}\}, &(104)\\
\mathcal{C}\ell_{1,3}^{\perp} &= \text{span}\,\{\mathfrak{n}^0, \mathfrak{n}^{01}, \mathfrak{n}^{02}, \mathfrak{n}^{03}, \mathfrak{n}^{012}, \mathfrak{n}^{023}, \mathfrak{n}^{031}, \mathfrak{n}^{0123}\}. &(105)
\end{aligned}$$

The 1-form fields $\mathfrak{n}^i \in T_x^*\Sigma$ are exactly the fields $\gamma_{\|}^i$ defined by eq.(58). As in the case of the α-grading given by $\alpha(\psi) = n\psi n^{-1}$ the parallel component $\mathcal{C}\ell_{1,3}^{\|}$ of $\mathcal{C}\ell_{1,3}$ is given by $\mathcal{C}\ell_{1,3}^{\|} \simeq \mathcal{C}\ell_{0,3} \simeq \mathbb{H}\oplus\mathbb{H}$, and then it is immediate to see that the spinor field composing eq.(103) is generated by $\{\mathtt{P}_\pm, \mathfrak{n}^{ij}\mathtt{P}_\pm\}$, where $\mathtt{P}_\pm = \frac{1}{2}(1 \pm \mathfrak{n}^{123})$ are idempotents spanned by central elements of $\mathcal{C}\ell_{0,3}$. Each copy $\mathbb{H}$ of $\mathbb{H}\oplus\mathbb{H} \simeq \mathcal{C}\ell_{0,3}$ is respectively generated by $\{\mathtt{P}_+, \mathfrak{n}^{ij}\mathtt{P}_+\}$ and $\{\mathtt{P}_-, \mathfrak{n}^{ij}\mathtt{P}_-\}$, and therefore the spinorial field $\psi \in \mathcal{C}\ell_{1,3}^{\|}$ with respect to the α-grading given by $\alpha(\psi) = n\hat{\psi}n^{-1}$ has the more general form given by

$$\psi = a\mathtt{P}_+ + b_{ij}\mathfrak{n}^{ij}\mathtt{P}_+ + c\mathtt{P}_- + d_{ij}\mathtt{P}_- \;\in \mathbb{H}\oplus\mathbb{H} \simeq \mathcal{C}\ell_{0,3} = \mathcal{C}\ell_{1,3}^{\|}, \tag{106}$$

where a, b_{ij}, c, d_{ij} are scalar functions with values in $\mathbb{C}$.

The possible values of $\sigma \in \mathcal{C}\ell_{1,3}^{||}$ are given by the conditions in eq.(102), and according to the elements in $\mathcal{C}\ell_{1,3}^{||}$ given by eq.(104) it follows immediately that

$$\sigma = a_3\mathfrak{n}^{12} + a_1\mathfrak{n}^{23} + a_2\mathfrak{n}^{31} \in \mathrm{SU}(2), \tag{107}$$

where $a_i \in \mathbb{C}$, since from the condition $\sigma^2 = -1$ we have $a_1^2 + a_2^2 + a_3^2 = 1$. Therefore $\sigma \in \mathrm{SU}(2)$ is a unitary quaternion.

The parallel and orthogonal projection operators acting on the Dirac operator, in the case when the α-grading is given by $\alpha(\psi)_n = n\hat{\psi}n^{-1}$ are defined by [11]

$$\pi_{||}(\partial) = \pi_{||}(\mathfrak{n}^\mu\partial_\mu) := \pi_{||}(\mathfrak{n}^\mu)\partial_\mu, \qquad \pi_\perp(\partial) = \pi_\perp(\mathfrak{n}^\mu\partial_\mu) := \pi_\perp(\mathfrak{n}^\mu)\partial_\mu. \tag{108}$$

We emphasize that $\mathfrak{n}^0 \equiv n$. Then eq.(103) can be written as

$$\mathfrak{n}^k\partial_k\psi n\sigma + n\partial_n\psi\sigma + m\psi n = 0. \tag{109}$$

Right multiplying the equation above by n it follows that

$$\mathfrak{n}^k\partial_k\psi\sigma + n\partial_n\psi n\sigma + m\psi = 0, \tag{110}$$

which is led to

$$\mathfrak{n}^k\partial_k\psi + \partial_n\hat{\psi} = m\psi\sigma \tag{111}$$

This is the Dirac equation with respect to the α-grading given by $\alpha(\psi) = n\hat{\psi}n^{-1}$, and motivated by spacetime decomposition. Spinor fields satisfying eq.(111) are given by eq.(106).

13 Concluding Remarks

An arbitrary Clifford algebra (AC) is split in α-even and α-odd components, related to a given inner automorphic α-grading, besides describing various consequences of this decomposition in the splitting of operators acting on the exterior and Clifford algebras. black hole. We use arbitrary AC α-gradings to generalize spacetime splitting in a superposition of infinite spacelike slices, each one in a fixed time. Such a decomposition consists of a local spacetime foliation. Based on a $\mathbb{Z}_2$-grading, induced by a inner automorphism of $\mathcal{C}\ell_{p,q}$, we decompose Dirac operator in parallel and orthogonal components, showing how each of these components is related to the Lie derivative along the splitting vector. Parallel and orthogonal projections are also introduced via inner automorphisms, induced by multivectorial fields. In this form we introduce another α-grading completely dual and equivalent to the α-grading $\alpha(\psi) = n\hat{\psi}n^{-1}$, $\psi \in \mathcal{C}\ell_{p,q}$, initially chosen. We have shown such dual α-grading is equivalent to the spacetime decomposition, but now in terms of the volume element associated with the α-even subalgebra. Dirac operator has been calculated in each of these dual decompositions. Multivectorial-induced decompositions are calculated for all elements of $\mathcal{C}\ell_{1,3}$. It can be shown that several symmetry breaking mechanisms in quantum field theory can be emulated using our formalism, such as $\mathfrak{so}(7) \mapsto \mathfrak{so}(6) \mapsto \mathfrak{su}(3) \times \mathfrak{u}(1)$.

The Dirac equation for an α-grading, associated with spacetime splitting via inner automorphisms, has been presented in a concise manner. Besides showing the Dirac spinor field is described by the direct sum of two quaternions, we have written Dirac equation, comparing our results with those recently obtained, also based on α-gradings induced by the standard, chiral (Weyl) and Majorana representations of $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ in $\mathcal{M}(4,\mathbb{C})$ [11].

We can show that relativistic kinematics and the evolution laws for the mass and spin of a test particle in the neighborhood of a Reissner-Nordstrøm black hole are described via the formalism of AC splittings. The next step is to obtain these laws for a Kerr Black hole, extending all field equations — based on Gauss-Codazzi equations and in the relation between AdS_5 bulk geometry and the geometry of the 3-brane [18] — and formalizing these concepts to investigate physical effects of extra dimensions [19], in a sequel paper.

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